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Decoherence of entanglement in the Bloch channel

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Abstract

Decoherence of entanglement of qubits is investigated which is caused by the phenomenological quantum channel (the Bloch channel) equivalent to the Bloch equations. It is shown how the decoherence of entanglement depends on the longitudinal and transverse relaxation times and the equilibrium value of the qubit. The quantum dense coding system under the influence of the Bloch channel is also investigated. The Shannon mutual information obtained by the Bell measurement and the Holevo capacity is calculated. Furthermore, the microscopic system-reservoir model which yields the Bloch equations is considered. The result shows that the temperature of the thermal reservoir significantly affects the decoherence of entanglement.

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1. Introduction

Quantum information processing has recently attracted much attention in quantum physics and information science. It provides the novel information technologies of quantum cryptography, quantum communication and quantum computation as well as the new insights on the principles of quantum mechanics [1, 2]. Entanglement between quantum systems is one of the most important resources in quantum information processing. When quantum information processing is performed in the real world, decoherence (or a relaxation process) caused by an external environment (or a thermal reservoir) is inevitable. This is the most serious obstacle in performing quantum communication and quantum computation with high performance. Therefore, one of the most important problems is to investigate how the decoherence affects the entanglement in quantum information processing.

The decoherence can be investigated by means of phenomenological methods, stochastic methods and microscopic methods [3–10]. In the microscopic approach, the interaction between the relevant system and the external environment is modelled and the projection

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operator method is applied to eliminate the variables of the external environment. In the stochastic approach, the effect of the external environment is treated as a stochastic process. In the phenomenological approach, parameters such as the relaxation times are introduced for describing the effect of the external environment. The phenomenological approach makes clear which parameters are essential for the decoherence of the relevant properties since the phenomenological parameters can be treated as independent variables. A typical example is the Bloch equations for investigating spin relaxation processes [11, 12]. The Bloch equations have widely been used for explaining various kinds of experimental data [13, 14]. Although the Bloch equations are very simple, they provide useful information about relaxation processes. Therefore the paper investigates the decoherence of entanglement of qubits (spin-1/2 systems) that is caused by the phenomenological quantum channel which is equivalent to the Bloch equations.

This paper is organized as follows. In section 2 we derive the quantum channel associated with the Bloch equations which have phenomenological longitudinal and transverse relaxation times [11, 12], and investigate the basic properties. The complete positivity of the quantum channel restricts the values of the relaxation times. In section 3 using the entanglement of formation, we consider the decoherence of entanglement of qubits caused by the Bloch channel, and obtain the condition that the Bloch channel becomes an entanglement-breaking channel [15, 16]. In section 4 we use the result for investigating the transmission of classical information in the quantum dense coding system of qubits [17–22]. The Holevo capacity [23, 24] is calculated and compared with the Shannon mutual information obtained by the Bell measurement. In section 5 we consider the microscopic system-reservoir model that yields the quantum channel equivalent to the Bloch equations, and obtain the microscopic expressions of the relaxation times. The temperature dependence of the decoherence of entanglement is found. In section 6 we give the concluding remarks.

2. Quantum channel associated with the Bloch equations

The Bloch equations proposed for investigating spin relation processes [11, 12] can also describe the decoherence of qubits, where they can be expressed in terms of the average values of the Pauli matrices as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_x\rangle_t = -\frac{1}{T_2}\langle\hat{\sigma}_x\rangle_t \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_y\rangle_t = -\frac{1}{T_2}\langle\hat{\sigma}_y\rangle_t \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle_t = -\frac{1}{T_1}(\langle\hat{\sigma}_z\rangle_t - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) \tag{3}$$

where T_1 and T_2 are the longitudinal and transverse relaxation times, and $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}$ is the equilibrium value of $\langle\hat{\sigma}_z\rangle_t$, namely, $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = \langle\hat{\sigma}_z\rangle_{t=\infty}$. In equations (1) and (2), we have ignored the angular frequency since it is not important for our purpose. Any quantum state $\hat{\rho}(t)$ of a single qubit subject to the Bloch equations is obtained in the following form:

$$\rho(t) = \tfrac{1}{2}[1 + a_x(t)\hat{\sigma}_x + a_y(t)\hat{\sigma}_y + a_z(t)\hat{\sigma}_z] \tag{4}$$

where the Bloch vector $\vec{a}(t) = (a_x(t), a_y(t), a_z(t))^{\mathrm{T}}$ is given by

$$a_x(t) = \langle\hat{\sigma}_x\rangle_t = \mathrm{e}^{-t/T_2}\langle\hat{\sigma}_x\rangle_0 \tag{5}$$

$$a_y(t) = \langle\hat{\sigma}_y\rangle_t = \mathrm{e}^{-t/T_2}\langle\hat{\sigma}_y\rangle_0 \tag{6}$$

$$a_z(t) = \langle\hat{\sigma}_z\rangle_t = \mathrm{e}^{-t/T_1}\langle\hat{\sigma}_z\rangle_0 + (1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}.. \tag{7}$$

Thus the Bloch equations completely determine the time evolution of any qubit state.

The quantum channel $\mathcal{L}_t$ defined by the relation $\hat{\rho}(t) = \hat{\mathcal{L}}_t\hat{\rho}(0)$, which we refer to as the Bloch channel, is determined by

$$\hat{\mathcal{L}}_t|0\rangle\langle 0| = \tfrac{1}{2}(1+\mathrm{e}^{-t/T_1})|0\rangle\langle 0| + \tfrac{1}{2}(1-\mathrm{e}^{-t/T_1})|1\rangle\langle 1| + \tfrac{1}{2}(1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|0\rangle\langle 0| - |1\rangle\langle 1|) \tag{8}$$

$$\hat{\mathcal{L}}_t|1\rangle\langle 1| = \tfrac{1}{2}(1-\mathrm{e}^{-t/T_1})|0\rangle\langle 0| + \tfrac{1}{2}(1+\mathrm{e}^{-t/T_1})|1\rangle\langle 1| + \tfrac{1}{2}(1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|0\rangle\langle 0| - |1\rangle\langle 1|) \tag{9}$$

$$\hat{\mathcal{L}}_t|0\rangle\langle 1| = \mathrm{e}^{-t/T_2}|0\rangle\langle 1| \tag{10}$$

$$\hat{\mathcal{L}}_t|1\rangle\langle 0| = \mathrm{e}^{-t/T_2}|1\rangle\langle 0| \tag{11}$$

where $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$. These equations can be derived by substituting $\vec{a}(0) = (0,0,0), (1,0,0), (0,1,0), (0,0,1)$ into the identity $\hat{\rho}(t) = \hat{\mathcal{L}}_t\hat{\rho}(0)$ with equations (4)–(7). The Bloch channel can be written in the form of the Pauli channel

$$\hat{\mathcal{L}}_t\hat{X} = p_0(t)\hat{X} + \sum_{k=x,y,z} p_k(t)\hat{\sigma}_k\hat{X}\hat{\sigma}_k \tag{12}$$

with

$$p_0(t) = \tfrac{1}{4}(1+2\,\mathrm{e}^{-t/T_2}+\mathrm{e}^{-t/T_1}) + \tfrac{1}{4}(1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \tag{13}$$

$$p_z(t) = \tfrac{1}{4}(1-2\,\mathrm{e}^{-t/T_2}+\mathrm{e}^{-t/T_1}) + \tfrac{1}{4}(1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \tag{14}$$

$$p_x(t) = p_y(t) = \tfrac{1}{4}(1-\mathrm{e}^{-t/T_1})(1-\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}). \tag{15}$$

In particular, when $T_1 = T_2$ and $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0$, the Bloch channel becomes the depolarizing channel. The quantum master equation (the Lindblad equation) equivalent to the Bloch equations is given by

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho}(t) &= \frac{1+\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}}{4T_1}([\hat{\sigma}_+,\hat{\rho}(t)\hat{\sigma}_-] + [\hat{\sigma}_+\hat{\rho}(t),\hat{\sigma}_-]) + \frac{1-\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}}{4T_1}([\hat{\sigma}_-,\hat{\rho}(t)\hat{\sigma}_+] \\ &\quad + [\hat{\sigma}_-\hat{\rho}(t),\hat{\sigma}_+]) + \frac{1}{4}\left(\frac{1}{2T_1} - \frac{1}{T_2}\right)[\hat{\sigma}_z,[\hat{\sigma}_z,\hat{\rho}(t)]] \\ &\equiv \hat{\mathcal{K}}\hat{\rho}(t)\end{aligned} \tag{16}$$

where the relation $\hat{\mathcal{L}}_t = \exp(\hat{\mathcal{K}}t)$ holds.

An any quantum channel must be completely positive [25]. This restricts the values of the longitudinal and transverse relaxation times T_1 and T_2. It is found from equations (8)–(11) that the Bloch channel $\hat{\mathcal{L}}_t$ is equivalent to the transformation of the Bloch vector

$$\vec{a}(t) = \mathsf{L}_t\vec{a}(0) + \vec{b}(t) \tag{17}$$

with

$$\mathsf{L}_t = \begin{pmatrix} \mathrm{e}^{-t/T_2} & 0 & 0 \\ 0 & \mathrm{e}^{-t/T_2} & 0 \\ 0 & 0 & \mathrm{e}^{-t/T_1}\end{pmatrix} \tag{18}$$

$$\vec{b}(t) = \begin{pmatrix} 0 \\ 0 \\ (1-\mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}\end{pmatrix}. \tag{19}$$

Then the condition for the Bloch channel $\hat{\mathcal{L}}_t$ to be completely positive is that the following inequality should be satisfied for an arbitrary time t [26]:

$$1 + \mathrm{e}^{-t/T_2} \geqslant 2\,\mathrm{e}^{-t/T_2}. \tag{20}$$

It is seen that this condition is equivalent to having the longitudinal relaxation time T_1 and the transverse relation time T_2 satisfy the inequality

$$2T_1 \geqslant T_2. \tag{21}$$

Since a completely positive map is positive, this inequality ensures the positivity of the quantum state $\hat{\rho}(t) = \hat{\mathcal{L}}_t\hat{\rho}(0)$, that is, $|\vec{a}(t)| \leqslant 1$. It is obvious that the relaxation times T_1 and T_2 in the Bloch equations derived from the microscopic system-reservoir model satisfy condition (21) (see section 5).

3. Decoherence of entanglement of formation

To investigate the decoherence of entanglement of qubits that is caused by the Bloch channel $\hat{\mathcal{L}}_t$, we suppose that one of the two qubits in the Bell state $|\Phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is sent through the Bloch channel $\hat{\mathcal{L}}_t$. Then the output state $\hat{\rho}_t$ of the Bloch channel $\hat{\mathcal{L}}_t$ becomes

$$\begin{aligned}\hat{\rho}_t &= (\hat{\mathcal{L}}_t \otimes \hat{\mathcal{I}})|\Phi_+\rangle\langle\Phi_+| \\ &= \tfrac{1}{4}(1 + 2\,\mathrm{e}^{-t/T_2} + \mathrm{e}^{-t/T_1})|\Phi_+\rangle\langle\Phi_+| + \tfrac{1}{4}(1 - 2\,\mathrm{e}^{-t/T_2} + \mathrm{e}^{-t/T_1})|\Phi_-\rangle\langle\Phi_-| \\ &\quad + \tfrac{1}{4}(1 - \mathrm{e}^{-t/T_1})(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \\ &\quad + \tfrac{1}{4}(1 - \mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\mathrm{h.c.}\})\end{aligned} \tag{22}$$

where $|\Phi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ and 'h.c.' stands for the Hermitian conjugate. The Hermitian operator $\hat{\rho}'_t = (\hat{\sigma}_y \otimes \hat{\sigma}_y)\hat{\rho}^*(\hat{\sigma}_y \otimes \hat{\sigma}_y)$ is calculated to be

$$\begin{aligned}\hat{\rho}'_t &= \tfrac{1}{4}(1 + 2\,\mathrm{e}^{-t/T_2} + \mathrm{e}^{-t/T_1})|\Phi_+\rangle\langle\Phi_+| + \tfrac{1}{4}(1 - 2\,\mathrm{e}^{-t/T_2} + \mathrm{e}^{-t/T_1})|\Phi_-\rangle\langle\Phi_-| \\ &\quad + \tfrac{1}{4}(1 - \mathrm{e}^{-t/T_1})(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \\ &\quad - \tfrac{1}{4}(1 - \mathrm{e}^{-t/T_1})\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\mathrm{h.c.}\}).\end{aligned} \tag{23}$$

The concurrence C_t of the quantum state $\hat{\rho}_t$ [27, 28] is given by

$$C_t = \max\left[0, 2\max_{1\leqslant k\leqslant 4}\lambda_k - \sum_{k=1}^{4}\lambda_k\right] \tag{24}$$

where λ_k $(1 \leqslant k \leqslant 4)$ is the eigenvalue of the Hermitian matrix $\hat{R} = (\sqrt{\hat{\rho}_t}\hat{\rho}'_t\sqrt{\hat{\rho}_t})^{1/2}$, which is equal to the square root of the eigenvalue of $\hat{\rho}_t\hat{\rho}'_t$. Then we find that

$$\lambda_1 = \tfrac{1}{4}\left[\sqrt{(1 + \mathrm{e}^{-t/T_1})^2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2(1 - \mathrm{e}^{-t/T_1})^2} + 2\,\mathrm{e}^{-t/T_2}\right] \tag{25}$$

$$\lambda_2 = \tfrac{1}{4}\left[\sqrt{(1 + \mathrm{e}^{-t/T_1})^2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2(1 - \mathrm{e}^{-t/T_1})^2} - 2\,\mathrm{e}^{-t/T_2}\right] \tag{26}$$

$$\lambda_3 = \tfrac{1}{4}\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}(1 - \mathrm{e}^{-t/T_1}) \tag{27}$$

$$\lambda_4 = \tfrac{1}{4}\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}(1 - \mathrm{e}^{-t/T_1}) \tag{28}$$

where we have used inequality (21) in deriving the eigenvalue λ_2. Hence we obtain the concurrence C_t of the quantum state $\hat{\rho}_t$

$$C_t = \max\left[0, \mathrm{e}^{-t/T_2} - \tfrac{1}{2}\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}(1 - \mathrm{e}^{-t/T_1})\right]. \tag{29}$$

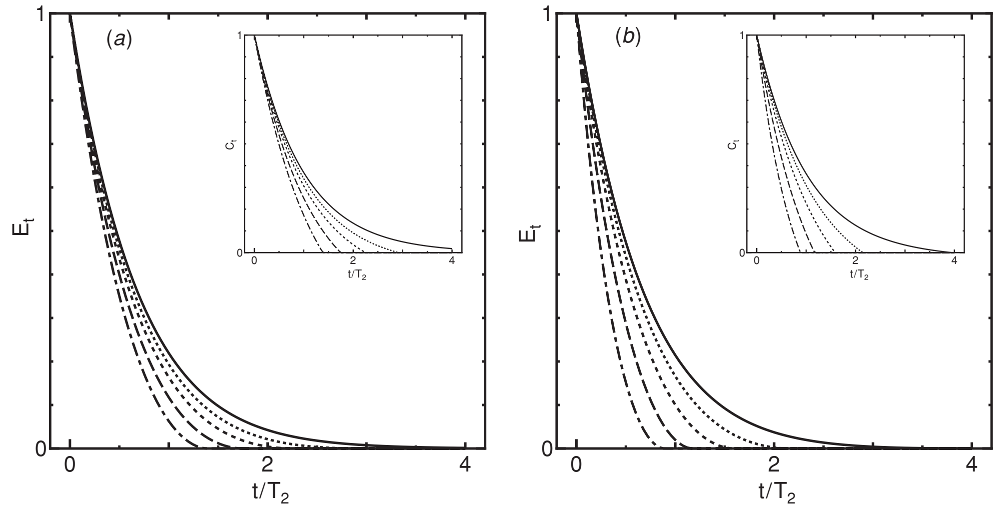

Figure 1. The time evolution of the entanglement of formation of the quantum state $\hat{\rho}_t$ under the influence of the Bloch channel, where (*a*) $T_1/T_2 = 2.0$; $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0$ (dash-dotted line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.8$ (dashed line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.95$ (short-dashed line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.99$ (dotted line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 1.0$ (solid line) and (*b*) $T_1/T_2 = 0.5$ (dash-dotted line), $T_1/T_2 = 1.2$ (dashed line), $T_1/T_2 = 3.0$ (short-dashed line), $T_1/T_2 = 8.0$ (dotted line), $T_1/T_2 = 100.0$ (solid line); $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.1$. The inset graphs show the concurrence C_t of the quantum state $\hat{\rho}_t$ for the same parameters.

The entanglement of formation E_t [27–29] is given by

$$E_t = \mathcal{F}(C_t) \tag{30}$$

with

$$\mathcal{F}(x) = H\left(\frac{1+\sqrt{1-x^2}}{2}\right) \tag{31}$$

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x). \tag{32}$$

The entanglement of formation E_t is plotted as a function of time in figure 1.

The necessary and sufficient condition for the quantum state $\hat{\rho}_t$ to be separable is obtained from equations (29) and (30)

$$\sqrt{1-\langle\hat{\sigma}_z\rangle_{\rm eq}^2}\,(1-{\rm e}^{-t/T_1}) \geqslant 2{\rm e}^{-t/T_2}. \tag{33}$$

This condition can also be derived from the positivity of the partial transposition of the quantum state $\hat{\rho}_t$ [30, 31]. From equations (29) and (33), we find the following results:

(i) if $\langle\hat{\sigma}_z\rangle_{\rm eq} = \pm 1$, the quantum state $\hat{\rho}_t$ is always entangled for finite time t and the concurrence C_t decays as $C_t = {\rm e}^{-t/T_2}$. In this case the relaxation time of the concurrence C_t is identical with the transverse relaxation time T_2. Note that $\langle\hat{\sigma}_z\rangle_{\rm eq} = \pm 1$ implies that the equilibrium state of the qubit is $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$;

(ii) if the longitudinal relation time T_1 is sufficiently large, the quantum state $\hat{\rho}_t$ is always entangled in the time region of $t \ll T_1$. In this case, the concurrence of the quantum state $\hat{\rho}_t$ is approximated with $C_t \approx {\rm e}^{-t/T_2}$, and thus the relaxation time of concurrence C_t is equal to the transverse relation time T_2;

(iii) if the transverse relaxation time T_2 is sufficiently short and $\langle\hat{\sigma}_z\rangle_{\rm eq} \neq \pm 1$, the quantum state $\hat{\rho}_t$ is always separable in the time region of $t \gg T_2$.

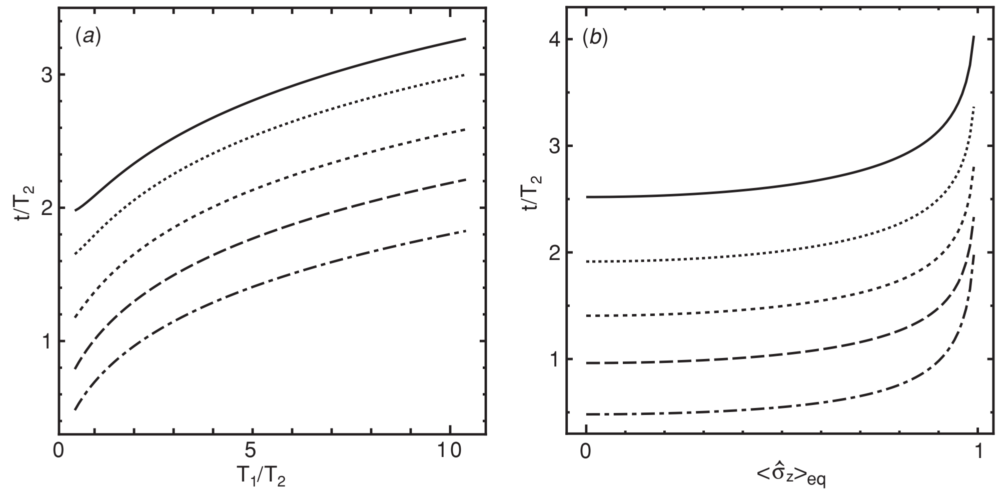

Figure 2. The time t when the quantum state $\hat{\rho}_t$ changes from entangled to separable under the influence of the Bloch channel, where (*a*) $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.0$ (dash-dotted line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.82$ (dashed line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.94$ (short-dashed line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.98$ (dotted line), $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.99$ (solid line) and (*b*) $T_1/T_2 = 0.5$ (dash-dotted line), $T_1/T_2 = 2.0$ (dashed line), $T_1/T_2 = 5.0$ (short-dashed line), $T_1/T_2 = 12.0$ (dotted line), $T_1/T_2 = 30.0$ (solid line). The quantum state $\hat{\rho}_t$ is separable in the time region upside of the each graph.

Note that when $(\hat{\mathcal{L}}_t \otimes \mathcal{I})|\Phi_+\rangle\langle\Phi_+|$ becomes separable, the quantum channel $\hat{\mathcal{L}}_t$ is an entanglement-breaking channel [15, 16]. Therefore, it is necessary for the Bloch channel $\hat{\mathcal{L}}_t$ to be an entanglement-breaking channel that the longitudinal relaxation time T_1 takes a finite value and the equilibrium value of $|\langle\hat{\sigma}_z\rangle_t|$ is less than unity. The time t when the quantum state $\hat{\rho}_t$ changes from entangled to separable under the influence of the Bloch channel or equivalently when the Bloch channel $\hat{\mathcal{L}}_t$ becomes an entanglement-breaking channel is shown in figure 2.

4. Information transmission by means of quantum dense coding

This section investigates the transmission of classical information by means of the quantum dense coding [17–22] under the influence of the Bloch channel. Considering the entanglement distillation, a sender (Alice) and a receiver (Bob) can share the Bell state $|\Phi_+\rangle$ even if a quantum channel is noisy [32]. Hence we suppose that Alice and Bob share the Bell state $|\Phi_+\rangle$. Alice encodes two bits of classical information by applying that of four operators $\hat{1}, \hat{\sigma}_z, \hat{\sigma}_x, \hat{\sigma}_y$ to her qubit. Then Alice sends the encoded qubit to Bob through the Bloch channel. After receiving it, Bob obtains one of the four two-qubit states

$$\begin{aligned}\hat{\rho}_{00} = {}& \tfrac{1}{4}(a_t + b_t)|\Phi_+\rangle\langle\Phi_+| + \tfrac{1}{4}(a_t - b_t)|\Phi_-\rangle\langle\Phi_-| + \tfrac{1}{4}c_t(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \\ & + \tfrac{1}{4}c_t\langle\hat{\sigma}_z\rangle_{\rm eq}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\text{h.c.}\})\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{\rho}_{01} = {}& \tfrac{1}{4}(a_t - b_t)|\Phi_+\rangle\langle\Phi_+| + \tfrac{1}{4}(a_t + b_t)|\Phi_-\rangle\langle\Phi_-| + \tfrac{1}{4}c_t(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \\ & + \tfrac{1}{4}c_t\langle\hat{\sigma}_z\rangle_{\rm eq}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\text{h.c.}\})\end{aligned} \tag{35}$$

$$\begin{aligned}\hat{\rho}_{10} = {}& \tfrac{1}{4}(a_t + b_t)|\Psi_+\rangle\langle\Psi_+| + \tfrac{1}{4}(a_t - b_2)|\Psi_-\rangle\langle\Psi_-| + \tfrac{1}{4}c_t(|\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-|) \\ & + \tfrac{1}{4}c_t\langle\hat{\sigma}_z\rangle_{\rm eq}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\text{h.c.}\})\end{aligned} \tag{36}$$

$$\hat{\rho}_{11} = \tfrac{1}{4}(a_t - b_t)|\Psi_+\rangle\langle\Psi_+| + \tfrac{1}{4}(a_t + b_t)|\Psi_-\rangle\langle\Psi_-| + \tfrac{1}{4}c_t(|\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-|)$$
$$+ \tfrac{1}{4}c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\mathrm{h.c.}\}) \quad (37)$$

where t is the transmission time of the encoded qubit and for the sake of simplicity we set

$$a_t = 1 + \mathrm{e}^{-t/T_1} \quad (38)$$
$$b_t = 2\mathrm{e}^{-t/T_2} \quad (39)$$
$$c_t = 1 - \mathrm{e}^{-t/T_1}. \quad (40)$$

Since the Holevo capacity C_{H} of the quantum dense coding system is attained when the prior probabilities are equal [22], we can obtain the Holevo capacity C_{H}

$$\begin{aligned} C_{\mathrm{H}} &= S\left(\frac{1}{4}\sum_{j,k=0,1}\hat{\rho}_{jk}\right) - \frac{1}{4}\sum_{j,k=0,1}S(\hat{\rho}_{jk}) \\ &= -\frac{1}{2}(1 + c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log_2(1 + c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) - \frac{1}{2}(1 - c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log_2(1 - c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) \\ &\quad + \frac{1}{2}c_t\log_2 c_t + \frac{1}{4}c_t(1 + \langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log_2(1 + \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) + \frac{1}{4}c_t(1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log_2(1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) \\ &\quad + \frac{1}{4}\left[a_t + \sqrt{b_t^2 + (c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})^2}\right]\log_2\left[a_t + \sqrt{b_t^2 + (c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})^2}\right] \\ &\quad + \frac{1}{4}\left[a_t - \sqrt{b_t^2 + (c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})^2}\right]\log_2\left[a_t - \sqrt{b_t^2 + (c_t\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})^2}\right]. \end{aligned} \quad (41)$$

In particular, if $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0$, the Holevo capacity is simplified as

$$C_{\mathrm{H}|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}=0} = \tfrac{1}{2}c_t\log_2 c_t + \tfrac{1}{4}(a_t + b_t)\log_2(a_t + b_t) + \tfrac{1}{4}(a_t - b_t)\log_2(a_t - b_t). \quad (42)$$

On the other hand, if the longitudinal relaxation times T_1 is sufficiently large and the condition $T_1 \gg t$ is satisfied, the Holevo capacity is approximated with

$$C_{\mathrm{H}|T_1\gg t} = 1 + \tfrac{1}{2}(1 + \mathrm{e}^{-t/T_2})\log_2(1 + \mathrm{e}^{-t/T_2}) + \tfrac{1}{2}(1 - \mathrm{e}^{-t/T_2})\log_2(1 - \mathrm{e}^{-t/T_2}). \quad (43)$$

When Bob performs the Bell measurement to extract the information encoded by Alice, the channel matrix [33] of the quantum dense coding system is given by

$$\mathsf{P}_{\mathrm{Bell}} = \frac{1}{4}\begin{pmatrix} a_t + b_t & a_t - b_t & c_t & c_t \\ a_t - b_t & a_t + b_t & c_t & c_t \\ c_t & c_t & a_t + b_t & a_t - b_t \\ c_t & c_t & a_t - b_t & a_t + b_t \end{pmatrix} \quad (44)$$

which does not depend on the equilibrium value $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}$ of the qubit. Since the Shannon mutual information becomes maximum when the prior probabilities are equal [34], the maximum value of the Shannon mutual information I_{Bell} is calculated to be

$$I_{\mathrm{Bell}} = \tfrac{1}{2}c_t\log_2 c_t + \tfrac{1}{4}(a_t + b_t)\log_2(a_t + b_t) + \tfrac{1}{4}(a_t - b_t)\log_2(a_t - b_t). \quad (45)$$

In particular, if the longitudinal relaxation times T_1 are sufficiently large and the condition $T_1 \gg t$ is satisfied, the Shannon mutual information I_{Bell} is approximated with

$$I_{\mathrm{Bell}|T_1\gg t} = 1 + \tfrac{1}{2}(1 + \mathrm{e}^{-t/T_2})\log_2(1 + \mathrm{e}^{-t/T_2}) + \tfrac{1}{2}(1 - \mathrm{e}^{-t/T_2})\log_2(1 - \mathrm{e}^{-t/T_2}). \quad (46)$$

Therefore, from equations (41)–(43), (45) and (46), we obtain the relations

$$C_{\mathrm{H}} \geqslant I_{\mathrm{Bell}} \qquad C_{\mathrm{H}|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}=0} = I_{\mathrm{Bell}} \qquad C_{\mathrm{H}|T_1\gg t} = I_{\mathrm{Bell}|T_1\gg t}. \quad (47)$$

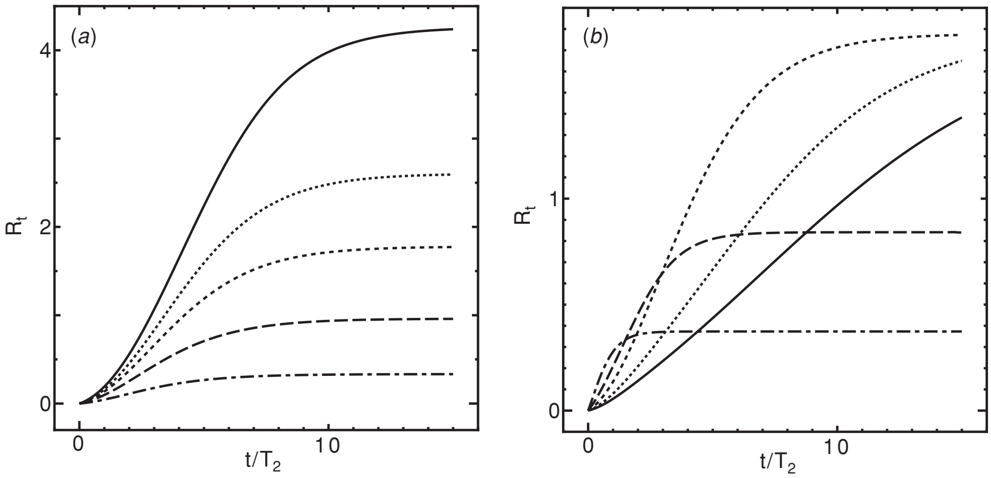

Figure 3. The time dependence of the ratio $R_t = (C_{\mathrm{H}} - I_{\mathrm{Bell}})/I_{\mathrm{Bell}}$, where (*a*) $T_1/T_2 = 2.0$; $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.5$ (dash-dotted line), $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.7$ (dashed line), $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.8$ (short-dashed line), $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.85$ (dotted line), $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.9$ (solid line) and (*b*) $T_1/T_2 = 0.5$ (dash-dotted line), $T_1/T_2 = 1.0$ (dashed line), $T_1/T_2 = 2.0$ (short-dashed line), $T_1/T_2 = 3.5$ (dotted line), $T_1/T_2 = 5.0$ (solid line); $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0.8$.

The fact that the Holevo capacity C_{H} is equal to the Shannon mutual information I_{Bell} implies that the quantum coding effect (or equivalently the super-additivity of the mutual information) [35–37] disappears. The ratio $R_t = (C_{\mathrm{H}} - I_{\mathrm{Bell}})/I_{\mathrm{Bell}}$ is plotted as a function of time in figure 3.

5. Microscopic system-reservoir model of the Bloch channel

We now consider the microscopic system-reservoir model of the decoherence that yields the Bloch channel under certain conditions. The system consists of a single spin of magnitude 1/2, namely, a qubit, and a thermal reservoir which is a set of harmonic oscillators in the thermal equilibrium with temperature T. The interaction Hamiltonian $\hat{H}_{\mathrm{int}}$ between the system and reservoir is assumed to be

$$\hat{H}_{\mathrm{in}} = \hbar \sum_k \left(g_k \hat{\sigma}_+ \hat{a}_k + g_k^* \hat{\sigma}_- \hat{a}_k^\dagger\right) \tag{48}$$

where $\hat{\sigma}_\pm = (1/2)(\hat{\sigma}_x \pm \mathrm{i}\hat{\sigma}_y)$, and $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the bosonic annihilation and creation operators of the kth mode, satisfying the canonical commutation relation $\left[\hat{a}_k, \hat{a}_l^\dagger\right] = \delta_{kl}$, and g_k is the coupling constant between the system and the reservoir. Applying the time-convolutionless formalism of the projection operator method [38–40] to eliminate the reservoir variables, we can obtain the quantum master equation of the qubit state $\hat{\rho}(t)$, up to the second order with respect to the coupling constant g_k, in the interaction picture,

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho}(t) = {}& \Phi_{+-}^*(t)[\hat{\sigma}_+, \hat{\rho}(t)\hat{\sigma}_-] + \Phi_{+-}(t)[\hat{\sigma}_+\hat{\rho}(t), \hat{\sigma}_-] \\ & + \Phi_{-+}^*(t)[\hat{\sigma}_-, \hat{\rho}(t)\hat{\sigma}_+] + \Phi_{-+}(t)[\hat{\sigma}_-\hat{\rho}(t), \hat{\sigma}_+]\end{aligned} \tag{49}$$

with

$$\Phi_{+-}(t) = \sum_k |g_k|^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \left\langle \hat{a}_k^\dagger(\tau)\hat{a}_k(0)\right\rangle_R \tag{50}$$

$$\Phi_{-+}(t) = \sum_k |g_k|^2 \int_0^t d\tau\, e^{i\omega\tau} \langle \hat{a}_k(\tau)\hat{a}_k^\dagger(0)\rangle_R \tag{51}$$

where $\hbar\omega$ is the energy separation between the upper and lower levels $|0\rangle$ and $|1\rangle$ of the qubit, and $\langle\cdots\rangle_R$ stands for the average value in the thermal equilibrium of the reservoir. By taking the Markovian limit (or equivalently the narrowing limit), the functions $\Phi_{+-}(t)$ and $\Phi_{-+}(t)$ are approximated as

$$\begin{aligned}\Phi_{+-}(t) &\approx \sum_k |g_k|^2 \int_0^\infty d\tau\, e^{-i\omega\tau} \langle \hat{a}_k^\dagger(\tau)\hat{a}_k(0)\rangle_R \\ &= \pi D(\omega)\bar{n}(\omega) - i\delta_+(\omega)\end{aligned} \tag{52}$$

$$\begin{aligned}\Phi_{-+}(t) &\approx \sum_k |g_k|^2 \int_0^\infty d\tau\, e^{i\omega\tau} \langle \hat{a}_k(\tau)\hat{a}_k^\dagger(0)\rangle_R \\ &= \pi D(\omega)[\bar{n}(\omega)+1] + i\delta_-(\omega)\end{aligned} \tag{53}$$

where the function $D(\omega)$ is the spectral density of the system-reservoir coupling and $\bar{n}(\omega)$ is the Bose–Einstein distribution

$$D(\omega) = \sum_k |g_k|^2 \delta(\omega_k - \omega) \tag{54}$$

$$\bar{n}(\omega) = (e^{\hbar\omega/k_B T} - 1)^{-1}. \tag{55}$$

Moreover the frequency shifts $\delta_\pm(\omega)$ are given by

$$\delta_+(\omega) = \mathrm{P}\int_0^\infty d\omega' \frac{1}{\omega-\omega'} D(\omega')\bar{n}(\omega') \tag{56}$$

$$\delta_-(\omega) = \mathrm{P}\int_0^\infty d\omega' \frac{1}{\omega-\omega'} D(\omega')[\bar{n}(\omega')+1] \tag{57}$$

where the symbol ‘P’ stands for taking the principal part of the integral. When the frequency shifts are negligible, we obtain the Markovian master equation for the qubit state [7]

$$\frac{\partial}{\partial t}\hat{\rho}(t) = R_\uparrow([\hat{\sigma}_+, \hat{\rho}(t)\hat{\sigma}_-] + [\hat{\sigma}_+\hat{\rho}(t), \hat{\sigma}_-]) + R_\downarrow([\hat{\sigma}_-, \hat{\rho}(t)\hat{\sigma}_+] + [\hat{\sigma}_-\hat{\rho}(t), \hat{\sigma}_+]) \tag{58}$$

with $R_\uparrow = \pi D(\omega)\bar{n}(\omega)$ and $R_\downarrow = \pi D(\omega)[\bar{n}(\omega)+1]$.

Comparing equation (58) with equation (16), we find the microscopic expressions for the longitudinal relaxation time T_1 and the transverse relaxation time T_2 and the equilibrium value w of $\langle\hat{\sigma}_z\rangle_t$

$$T_1 = \frac{1}{2(R_\uparrow + R_\downarrow)} = \frac{1}{2\pi D(\omega)[2\bar{n}(\omega)+1]} \tag{59}$$

$$T_2 = \frac{1}{R_\uparrow + R_\downarrow} = \frac{1}{\pi D(\omega)[2\bar{n}(\omega)+1]} \tag{60}$$

$$\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = \frac{R_\uparrow - R_\downarrow}{R_\uparrow + R_\downarrow} = -\frac{1}{2\bar{n}(\omega)+1} = -\tanh\left(\frac{1}{2}\beta\hbar\omega\right) \tag{61}$$

where $\beta = 1/k_B T$ with T being the temperature of the thermal reservoir. In this case, the equality $2T_1 = T_2$ holds in equation (21). Furthermore the longitudinal relaxation time T_1 and the transverse relaxation time T_2 can be expressed in terms of the equilibrium value w of $\langle\hat{\sigma}_z\rangle_t$

$$T_1 = -T_c\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \qquad T_2 = -2T_c\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \tag{62}$$

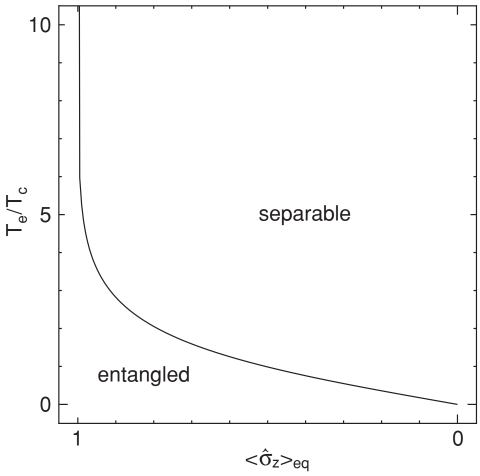

Figure 4. The dependence of the threshold time T_e on the equilibrium value $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}$. The quantum state $\hat{\rho}_t$ is separable (or entangled) in the upper (or lower) time region.

with $T_c = 1/2\pi D(\omega)$. Here it should be noted that the inequality $-1 \leqslant \langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \leqslant 0$ holds. Then the necessary and sufficient condition of the separability of the quantum state $\hat{\rho}_t = (\hat{\mathcal{L}}_t \otimes \hat{\mathcal{I}})|\Phi_+\rangle\langle\Phi_+|$ given by equation (33) becomes

$$\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\,(1 - \mathrm{e}^{-t/T_c|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}|}) \geqslant 2\mathrm{e}^{-t/2T_c|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}|}. \tag{63}$$

Hence the quantum state $\hat{\rho}_t$ is separable for $t \geqslant T_e$ and entangled for $t < T_e$, where the threshold time T_e is given by

$$T_e = 2T_c|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}| \ln\left(\frac{\sqrt{2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2} + 1}{\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}}\right). \tag{64}$$

It is easy to see that $\lim_{T\to 0} T_e = \infty$ and $\lim_{T\to\infty} T_e = 0$. The result implies that the quantum state $\hat{\rho}_t$ is always entangled when the thermal reservoir is in the vacuum state while it is always separable when the temperature of the environment is sufficiently high. The threshold time T_e is plotted as a function of w in figure 4. The figure clearly shows that the threshold time T_e rapidly decreases as the temperature T increases.

6. Conclusion

We have investigated the properties of the decoherence of entanglement that is caused by the phenomenological Bloch channel. We have shown how the decoherence of entanglement depends on the longitudinal and transverse relaxation times and the equilibrium value of the qubit. We have obtained the condition that the Bloch channel becomes an entanglement-breaking channel. Using the results, we have investigated the transmission of classical information by means of quantum dense coding under the influence of the Bloch channel. We have calculated the Shannon mutual information by the Bell measurement and the Holevo capacity of the quantum dense coding system. We have found that the Holevo capacity is equal to the Shannon mutual information under certain condition. This implies that the super-additivity of the mutual information (or equivalently the quantum coding effect) in the quantum dense coding system disappears. Furthermore using the microscopic system-reservoir model, we have obtained the microscopic expressions of the longitudinal and transverse relaxation times and the equilibrium value of the qubit. This result shows that the temperature of the thermal reservoir significantly affects the decoherence of entanglement.

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